\(\alpha\) -CONNECTEDNESS BETWEEN SOFT SETS

Alpa Singh Rajput\(^1\), S. S. Thakur\(^2\)

\(^{1}\)Assistant Professor, Science and Humanities Department Vignan Foundation for Science, Technology & Research University, India; \(^{2}\)Principal, Jabalpur Engineering College, Jabalpur, India.

Email: \(^{1}\)alpasinghrajput09@gmail.com, \(^{2}\)samajh_singh@rediffmail.com

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Abstract

Purpose of the study: In the present paper the concept of soft \(\alpha\) -connectedness between soft sets in soft topological spaces has been introduced and studied. The notion of connectedness captures the idea of hanging-togetherness of image elements in an object by giving a firmness of connectedness to every feasible path between every possible pair of image elements. It is an important tool for the designing of algorithms for image segmentation. The purpose of this paper is to extend the concept of \(\alpha\)–connectedness between sets in soft topology.

Main Findings: If a soft topological space (X, \(\tau\), E) is soft \(\alpha\) -connected between a pair of its soft sets, then it is not necessarily that it is soft \(\alpha\)–connected between each pair of its soft sets and so it is not necessarily soft \(\alpha\) -connected.

Applications of this study: Image Processing.

Novelty/Originality of this study: Extend of \(\alpha\) -connectedness between soft sets in soft topology.

Keywords: Soft a-open Sets, Soft a- Closed sets, Soft a-connectedness, Soft a-connectedness Between Soft Sets.

INTRODUCTION


Preliminaries

Let U is an initial universe set, E be a set of parameters, P(U) be the power set of U and A \(\subset E\).

**Definition 1:** (Molodtsov, D. (1999)), A pair (F, A) is called a soft set over U, where F is a mapping given by F: A \(\rightarrow P(U)\).

In other words, a soft set over U is a parameterized family of subsets of the universe U. For all \(e \in A\), F (e) may be considered as the set of e-approximate elements of the soft set (F, A).

**Definition 2:** (Maji, P. K. et al., (2003)), For two soft sets (F, A) and (G, B) over a common universe U, we say that (F, A) is a soft subset of (G, B) denoted by (F, A) \(\subset (G, B)\), if

(a) A \(\subset B\)
(b) F(e) \(\subset G(e)\) for all \(e \in E\).

**Definition 3:** (Maji, P. K. et al., (2003)) Two soft sets (F, A) and (G, B) over a common universe U are said to be soft equal denoted by (F, A) = (G, B) if (F, A) \(\subset (G, B)\) and (G, B) \(\subset (F, A)\).
Definition 4: (Majumdar, P. & Samanta, S. K. (2008)), The complement of a soft set \((F, A)\) denoted by \((F, A)^c\) is defined by \((F, A)^c = (F^c, A)\), where \(F^c: A \rightarrow P(U)\) is a mapping given by \(F^c(e) = U - F(e)\), for all \(e \in E\).

Definition 5: (Maji, P. K. et al., (2003)), Let a soft set \((F, A)\) over \(U\).

(a) Null soft set denoted by \(\emptyset\) if for all \(e \in A\), \(F(e) = \emptyset\).

(b) Absolute soft set denoted by \(U\), if for each \(e \in A\), \(F(e) = U\).

Clearly, \(U^c = \emptyset\) and \(\emptyset = U\).

Definition 6: (Ali, I. et al., (2009)), Union of two sets \((F, A)\) and \((G, B)\) over the common universe \(U\) is the soft \((H, C)\), where \(C = A \cup B\) and \(H(e) = F(e) \cup G(e)\) for each \(e \in C\).

Definition 7: (Ali, I. et al., (2009)), Intersection of two soft sets \((F, A)\) and \((G, B)\) over a common universe \(U\), is the soft set \((H, C)\) where \(C = A \cap B\) and \(H(e) = F(e) \cap G(e)\) for each \(e \in C\).

Definition 8: (Shabir, M. & Naz, M. (2011)), Let \(X\) and \(Y\) are an initial universe sets and \(E\) and \(K\) are the non-empty sets of parameters,

\[ S(X, E) \text{ denotes the family of all soft sets over } X, \text{ and } S(Y, K) \text{ denotes the family of all soft sets over } Y. \]

A subfamily \(\tau\) of \(S(X, E)\) is called a soft topology on \(X\) if:

(1) \(\emptyset, X\) belongs to \(\tau\).

(2) The union of any number of soft sets in \(\tau\) belongs to \(\tau\).

(3) The intersection of any two soft sets in \(\tau\) belongs to \(\tau\).

The triplet \((X, \tau, E)\) is called a soft topological space over \(X\). The members of \(\tau\) are called soft open sets in \(X\) and their complements are called soft closed sets in \(X\).

Definition 9: (Shabir, M. & Naz, M. (2011)), (Zorlutana, I. et al., (2012)), If \((X, \tau, E)\) is soft topological space and \((F, E) \in S(X, E)\).

(a) The closure of \((F, E)\) is denoted by \(Cl(F, E)\), which is defined as the intersection of all soft closed super sets of \((F, E)\).

(b) The interior of \((F, E)\) is denoted by \(Int(F, E)\), is defined as the union of all soft open subsets of \((F, E)\).

Definition 10: (Shabir, M. & Naz, M. (2011)), (Zorlutana, I. et al., (2012)), Let \((X, \tau, E)\) be a soft topological space and let \((F, E), (G, E) \in S(X, E)\).

(a) \((F, E)\) is soft closed if \((F, E) = Cl(F, E)\).

(b) If \((F, E) \subseteq (G, E)\), then \(Cl(F, E) \subseteq Cl(G, E)\).

(c) \((F, E)\) is soft open if \((F, E) = Int(F, E)\).

(d) If \((F, E) \subseteq (G, E)\), then \(Int(F, E) \subseteq Int(G, E)\).

(f) \((Int(F, E))^c = Cl((F, E))^c\).

Definition 11: (Zorlutana, I. et al., (2012)), The soft set \((F, E) \in S(X, E)\) is called a soft point if there exists \(x \in X\) and \(e \in E\) such that,

\[ F(e) = \{x\} \text{ and } F(e') = \emptyset \text{ for each } e' \in E - \{e\}, \text{ and the soft point } (F, E) \text{ is denoted by } \{xe\}_E. \]


(a) Soft \(\alpha\)-open if \((A, E) \subseteq Int(Cl(Int(A, E)))\).

(b) Soft semi-open if \((F, E) \subseteq Cl(Int(F, E))\).

(c) Soft pre-open if \((F, E) \subseteq int(Cl(F, E))\).
The complement of soft α-open set (resp. soft pre-open, soft semi-open) set is called soft α-closed (resp. soft pre-closed, soft semi-closed) set.

**Definition 13:** (Akdag, M. & Ozkan, A. (2014)) Let (F, E) be a soft set in a soft topological space (X, τ, E)

(a) The soft α-closure of (F, E) is defined as the smallest soft α-closed set over which contains (F, E), and it is denoted by α Cl(F, E).

(b) The soft α-interior of (F, E) is defined as the largest soft α-open set over which is contained in (F, E) and is denoted by α Int(F, E).

**Definition 14:** (Thakur, S. S. & Rajput, A. S. (2016), (2017), (2018), Hussain, S. (2015)), A soft topological space (X, τ, E) is said to be soft connectedness (resp. P-connected, s-connectedness) between soft sets (F1, E) and (F2, E) if and only if there is no soft clopen (resp. preclopen, semiopen) set (F, E) over X such that (F1, E) ⊂ (F, E) and (F, E) ∩ (F2, E) = Ø.

**Remark:** (Thakur, S. S. & Rajput, A. S. (2016), (2017), (2018)), A soft topological space (X, τ, E) is soft connected (resp. preconnected, semi-connected) if and only if it soft connected (resp. p-connected, s-connected) between every pair of its nonempty soft sets.

**α-connectedness between soft sets**
Throughout this paper soft α-clopen means soft α-closed-α-open.

**Definition 1:** A soft α-clopen set of soft topological space (X, τ, E) is a pair (F, E), (G, E) of disjoint non-null soft α-open sets whose union is X.

**Definition 2:** A soft topological space (X, τ, E) is said to be a soft α-connected, if there does not exist a soft α-separation of X.

**Definition 3:** A soft topological space (X, τ, E) is said to be soft α-connected between its soft subsets (F1, E) and (F2, E) if and only if there is no soft α-clopen subset (F, E) over X such that (F1, E) ⊂ (F, E) and (F, E) ∩ (F2, E) = Ø.

**Theorem 1:** A soft topological space (X, τ, E) is soft α-connected between soft sets (F1, E) and (F2, E) if and only if there is no soft α-clopen set (F, E) over X such that (F1, E) ⊂ (F, E) ⊂ (F2, E).

Proof: Follows from definition 3.3.

**Theorem 2:** If a soft topological space (X, τ, E) is soft α-connected between soft sets (F1, E) and (F2, E) then (F1, E) ≠ Ø ≠ (F2, E).

Proof: If any soft subset (F1, E) ≠ Ø, then Ø being soft α-clopen set over X, (X, τ, E) cannot be soft α-connected between soft sets (F1, E) and (F2, E).

**Theorem 3:** If a soft topological space (X, τ, E) is soft α-connected between soft sets (F1, E) and (F2, E) and (F1, E) ⊂ (F3, E) and (F2, E) ⊂ (F4, E) then (X, τ, E) is soft α-connected between soft sets (F3, E) and (F4, E).

Proof: Suppose soft topological space (X, τ, E) is not soft α-connected between soft sets (F3, E) and (F4, E) then there is a soft α-clopen set (F, E) over X such that (F3, E) ⊂ (F, E) and (F, E) ∩ (F4, E) = Ø. Consequently, (X, τ, E) is not soft α-connected between soft sets (F1, E) and (F2, E).

**Theorem 4:** A soft topological space (X, τ, E) is soft α-connected between soft sets (F1, E), and (F2, E) if and only if (X, τ, E) is soft α-connected between soft sets α Cl(F1, E) and α Cl(F2, E).

Proof: Necessity: Obviously.

Sufficiency: If soft topological space (X, τ, E) is not soft α-connected between soft sets (F1, E) and (F2, E), then there exists soft α-clopen set (F, E) over X such that (F1, E) ⊂ (F, E) and (F, E) ∩ (F2, E) ≠ Ø. Since (F, E) is soft α-closed, α Cl(F1, E) ⊂ α Cl(F, E) = (F, E), Clearly, (F, E) ∩ α Cl(F2, E) = Ø. For if (x) ∈ (F, E) ∩ α Cl(F2, E) then (F, E) ∩ (F2, E) ≠ Ø, because (F, E) is soft α-open. Hence, (X, τ, E) is not soft α-connected between soft sets α Cl(F1, E) and α Cl(F2, E).

**Theorem 5:** A soft topological space (X, τ, E) is not soft α-connected between (A0, E) and (A1, E) if and only if there exist soft α-clopen disjoint soft sets (F0, E) and (F1, E) such that X = (F0, E) U (F1, E) and (Ai, E) ⊂ (Fi, E), i = 0, 1.

Proof: This immediately follows from the definition of a space soft α-connected between two of its soft subsets.
Theorem 6: If \((F_1, E)\) and \((F_2, E)\) are soft sets in soft topological space \((X, \tau, E)\) and \((F_1, E) \cap (F_2, E) \neq \emptyset\), then \((X, \tau, E)\) is soft \(\alpha\) -connected between \((F_1, E)\) and \((F_2, E)\).

Proof: If \((F, E)\) is any soft \(\alpha\) -clopen set over \(X\) such that \((F_1, E) \subset (F, E)\), then \((F_1, E) \cap (F_2, E) \neq \emptyset \rightarrow (F, E) \cap (F_2, E) \neq \emptyset\). This proves the theorem.

Remark: The converse of Theorem 6 need not be true.

Example 1: Let \(X = \{x_1, x_2, x_3, x_4\}\) be universe set and \(E = \{e_1, e_2\}\) be the set of parameter. The soft sets \(\text{Let } (F, E), (F_1, E)\) and \((F_2, E)\) over \(X\) are defined as follows:

\(F(e_1) = \{x_1, x_3\}, F(e_2) = \{x_1, x_3\}\),
\(F_1(e_1) = \{x_1, x_3\}, F_1(e_2) = \{x_1, x_3\}\),
\(F_2(e_1) = \{x_2, x_4\}, F_2(e_2) = \{x_2, x_4\}\)

Let \(\tau = \{\emptyset, X, (F, E)\}\) be a soft topology on \(X\), then soft topological space \((X, \tau, E)\) is soft \(\alpha\) -connected between the soft sets \((F_1, E)\) and \((F_2, E)\) whereas, \((F_1, E) \cap (F_2, E) = \emptyset\).

Theorem 7: If a soft topological space \((X, \tau, E)\) is neither soft \(\alpha\) -connected between \((A, E)\) and \((B_0, E)\) nor soft \(\alpha\) -connected between \((A, E)\) and \((B_1, E)\) then it is not soft \(\alpha\) -connected between \((A, E)\) and \((B_0, E) \cup (B_1, E)\).

Proof: Since a soft topological space \((X, \tau, E)\) is not soft \(\alpha\) -connected between \((A, E)\) and \((B_0, E)\), there is a soft \(\alpha\) -clopen set \((F_0, E)\) over \(X\) such that \((A, E) \cap (F_0, E) = \emptyset\). Also since \((X, \tau, E)\) is not soft \(\alpha\) -connected between \((A, E)\) and \((B_1, E)\), there exists a soft \(\alpha\) -clopen set \((F_1, E)\) over \(X\) such that \((A, E) \cap (F_1, E) = \emptyset\). Put \((F, E) = (F_0, E) \cap (F_1, E)\). Since each soft \(\alpha\)-closed set is soft \(\alpha\)-closed and any intersection of soft \(\alpha\)-closed set is soft \(\alpha\)-closed, \((F, E)\) is soft \(\alpha\)-open. Therefore \((F, E)\) is soft \(\alpha\) -open over \(X\) such that \((A, E) \cap (F, E) = \emptyset\). Hence, \((X, \tau, E)\) is not soft \(\alpha\) -connected between \((A, E)\) and \((B_0, E) \cup (B_1, E)\).

Theorem 8: A soft topological space \((X, \tau, E)\) is soft \(\alpha\) -connected if and only if it is soft \(\alpha\) -connected between every pair of its nonempty soft sets.

Proof: Let \((A, E)\) and \((B, E)\) be a pair of nonempty soft sets over \(X\). Suppose \((X, \tau, E)\) is not soft \(\alpha\) -connected between \((A, E)\) and \((B, E)\). Then there is a soft \(\alpha\) -clopen set \((F, E)\) over \(X\) such that \((A, E) \cap (F, E) = \emptyset\). Since \((A, E)\) and \((B, E)\) are nonempty it follows that \((F, E)\) is a nonempty soft proper \(\alpha\) -clopen set over \(X\). Hence, \((X, \tau, E)\) is not soft \(\alpha\) -connected.

Conversely, suppose that \((X, \tau, E)\) is not soft \(\alpha\) -connected. Then there exists a nonempty proper soft set \((F, E)\) over \(X\) which is both soft \(\alpha\)-open and soft \(\alpha\)-closed. Consequently, \((X, \tau, E)\) is not soft \(\alpha\) -connected between \((F, E)\) and \((F, E)^C\). Thus, \((X, \tau, E)\) is not soft \(\alpha\) -connected between an arbitrary pair of its nonempty soft sets.

Remark: If a soft topological space \((X, \tau, E)\) is soft \(\alpha\) -connected between a pair of its soft sets, then it is not necessarily that it is soft \(\alpha\) -connected between each pair of its soft sets and so it is not necessarily soft \(\alpha\) -connected.

Example 2: Let \(X = \{h_1, h_2, h_3\}\) be an universe set, \(E = \{e_1, e_2\}\) be the set of parameter and the soft sets \((F_1, E), (F_2, E), (F_3, E), (F_4, E), (F_5, E)\) and \((F_6, E)\) over \(X\) are defined as follows:

\(F_1(e_1) = \{h_2\}, F_1(e_2) = \{h_1\}\),
\(F_2(e_1) = \{h_3\}, F_2(e_2) = \{h_2\}\),
\(F_3(e_1) = \{h_2, h_3\}, F_3(e_2) = \{h_1, h_2\}\),
\(F_4(e_1) = \{h_1, h_2\}, F_4(e_2) = X\),
\(F_5(e_1) = \{h_1, h_2\}, F_5(e_2) = \{h_1, h_3\}\),
\(F_6(e_1) = \emptyset, F_6(e_2) = \{h_2\}\),

Let \(\tau = \{\emptyset, X, (F_1, E), (F_2, E), (F_3, E), (F_4, E), (F_5, E), (F_6, E)\}\) be a soft topology over \(X\). Then the soft topological space \((X, \tau, E)\) is soft \(\alpha\) -connected between the soft sets \((F_3, E)\) and \((F_5, E)\) but it is not soft \(\alpha\) -connected between \((F_2, E)\) and \((F_5, E)\). Also, the soft topological space \((X, \tau, E)\) is not soft \(\alpha\) – connected.

Theorem 9: Every soft topological space which is soft \(s\) -connected between soft sets \((F_1, E)\) and \((F_2, E)\) is soft \(\alpha\) -connected between \((F_1, E)\) and \((F_2, E)\).
Proof: Suppose soft topological space \((X, \tau, E)\) is not soft \(\alpha\) -connected between \((F_1, E)\) and \((F_2, E)\). Then there is a soft \(\alpha\)-clopen set \((F, E)\) over \(X\) such that \((F_1, E) \cap (F, E) \cap (F_2, E) = \emptyset\). Since every soft \(\alpha\)-clopen set is soft semi-clopen, it follows that \((X, \tau, E)\) is not soft \(s\)-connected between \((F_1, E)\) and \((F_2, E)\). This is a contradiction.

**Remark:** The converse of theorem 9 may not be true.

**Example 3:** Let \(X = \{x_1, x_2, x_3\}\) be universe set and \(E = \{e_1, e_2\}\) be the set of parameter. The soft sets \(L(F, E), (F_1, E), (F_2, E), (F_3, E)\) and \((G, E)\) over \(X\) are defined as follows:
\[
F(e_1) = \{x_2\}, F(e_2) = \{x_3\}, F_1(e_1) = \{x_1\}, F_1(e_2) = \{x_2\}, F_2(e_1) = \{x_2\}, F_2(e_2) = \{x_2, x_3\}, G(e_1) = \emptyset, G(e_2) = \{x_2\},
\]

\(\) Let, \(\tau = \{\emptyset, X, (F_1, E), (F_2, E), (F_3, E)\}\) be a soft topology on \(X\). Then soft topological space \((X, \tau, E)\) is soft \(\alpha\) -connected between the soft sets \((F, E)\) and \((G, E)\) but not soft \(s\)-connectedness between \((F_1, E)\) and \((G, E)\).

**Theorem 10:** Every soft topological space which is soft \(P\)-connected between soft sets \((F_1, E)\) and \((F_2, E)\) is soft \(\alpha\) -connected between \((F_1, E)\) and \((F_2, E)\).

Proof: Suppose soft topological space \((X, \tau, E)\) is not soft \(\alpha\) -connected between \((F_1, E)\) and \((F_2, E)\). Then there is a soft \(\alpha\)-clopen set \((F, E)\) over \(X\) such that \((F_1, E) \subset (F, E) \subset (F_2, E) = \emptyset\). Since every soft \(\alpha\)-clopen set is soft pre-clopen, it follows that \((X, \tau, E)\) is not soft \(P\)-connected between \((F_1, E)\) and \((F_2, E)\). This is a contradiction.

**Remark:** The converse of theorem 3.19 may not be true.

**Example 4:** Let \(X = \{x_1, x_2, x_3, x_4\}\), \(E = \{e_1, e_1\}\) and soft sets are defined as : \((F, E) = \{(e_1,\{x_1, x_3\}), (e_2, \{x_1, x_3\})\}\), \((F_1, E) = \{(e_1,\{x_1\}), (e_2, \{x_1\})\}\) and \((F_2, E) = \{(e_1,\{x_3\}), (e_2, \{x_3\})\}\). Let \(\tau = \{\emptyset, (F, E), (F_1, E), (F_2, E), (X, E)\}\) is topology on \(X\). Then soft topological space \((X, \tau, E)\) is soft \(\alpha\) -connected between the soft sets \((F_1, E)\) and \((F_2, E)\) but not soft \(P\)-connectedness between \((F_1, E)\) and \((F_2, E)\).

Thus we reach the following diagram of implications.

![Diagram](https://example.com/diagram.png)

**CONCLUSION**

Connectedness is an important and major area of topology and it can give many relationships between other scientific areas and mathematical models. The notion of connectedness captures the idea of hanging-togetherness of image elements in an object by given a firmness of connectedness to every feasible path between every possible pair of image elements. It is an important tool for the designing of algorithms for image segmentation. Recently, many scientists have improved the soft set theory, which is introduced by Molodtsov and easily applied to many problems having uncertainties. In this paper, we introduced the notion of soft \(\alpha\) connectedness between soft sets in soft topological spaces. Soft \(\alpha\) connectedness is weaker than to soft semi connectedness and soft pre connectedness. It is shown that soft topological space is \(\alpha\) soft connected if and only if it is \(\alpha\) soft connected between every pair of its nonempty soft sets. I hope that the results established in this paper will help the researcher to enhance and promote the further study on the soft topology to carry out a general framework for the development of information systems.

**REFERENCES**
