\((\lambda,\mu)\)-Multi Anti Fuzzy subgroup of a group

Kr. Balasubramanian\(^1\), R. Revathy\(^2\)

\(^1\)Assistant Professor in Mathematics, H.H. The Rajah’s college(Affiliated to bharathidasan University, Trichy), Pudukkottai, Tamil Nadu, India; \(^2\)Research Scholar in Mathematics, H.H. The Rajah’s college(Affiliated to bharathidasan University, Trichy), Pudukkottai, Tamil Nadu, India.

Email: bbalamohith@gmail.com, revathyrsmaths@gmail.com

Abstract

**Purpose of the study:** To develop \((\lambda,\mu)\)-anti fuzzy subgroup of a group.

**Methodology:** The fundamental idea of \((\lambda,\mu)\)-anti fuzzy subgroup to create a \((\lambda,\mu)\)-multi anti fuzzy subgroup.

**Main Findings:** \((\lambda,\mu)\) – multi anti fuzzy cosets of a group.

**Applications of this study:** The advancement of the theory of a group’s multiple fuzzy subgroups.

**Novelty/Originality of this study:** The concept of \((\lambda,\mu)\)-multi anti fuzzy cosets of a group has been defined, and various associated theorems have been demonstrated using examples.

INTRODUCTION

Fuzzy sets were first introduced by [Feng, Y. and Yao, B. (2012)] and then the fuzzy sets have been used in the reconsideration of classical mathematics. [Yuan, X., Zhang, C., and Ren, Y. (2003)] introduced the concept of fuzzy subgroup with thresholds. A fuzzy subgroup with thresholds \(\lambda\) and \(\mu\) is also called a \(\textit{(}\lambda,\mu)\)-fuzzy subgroup. [Yao] continued to research \((\lambda,\mu)\)-fuzzy normal subgroups, \((\lambda,\mu)\)-fuzzy quotient subgroups and \((\lambda,\mu)\)-fuzzy subrings in [Yao, B. (2005)]. [Shen] researched anti-fuzzy subgroups in [Shen, Z (1995)]. By a fuzzy subset of a non-empty set \(X\) we mean a mapping from \(X\) to the unit interval \([0,1]\). Throughout this article, we will always assume that \(0 \leq \lambda < \mu \leq 1\). [Atanassov, K.T. (1986), Atanassov, K.T. (1994), Mukharjee, N. P. and Bhattacharya, P. (1984), Zadeh, L.A. (1965)]


Preliminaries Definition: 2.1 ([Feng, Y. and Yao, B. (2012)])

Let \(X\) be a non-empty set. A fuzzy subset \(A\) of \(X\) is defined by a function \(A:X\rightarrow[0,1]\).

**Definition: 2.2** ([Sabu, S. and Ramakrishnan, T.V.(2011a), Sabu, S., Ramakrishnan, T.V.(2011b)])

Let \(X\) be a non-empty set. A multi fuzzy set \(A\) in \(X\) is defined as the set of ordered sequences as follows. \(A = \{(x, A_1(x), A_2(x), \ldots, A_k(x), \ldots) : x \in X\}\). Where \(A_i:X \rightarrow [0,1] \ for \ all \ i\).

**Definition: 2.3** ([Sabu, S., Ramakrishnan, T.V.(2011b)])

Let \(X\) be a non-empty set. A \(k\)-dimensional multi fuzzy set \(A\) in \(X\) is defined by the set \(A = \{(x, (A_1(x), A_2(x), \ldots, A_k(x))) : x \in X\}\). Where \(A_i:X \rightarrow [0,1] \ for \ i = 1,2,3, \ldots, k\).

**Definition: 2.4** ([Feng, Y. and Yao, B.(2012)])

Let \(A\) be a fuzzy subset of \(G\). \(A\) is called a \((\lambda,\mu)\)-anti fuzzy subgroup of \(G\) if, for all \(x, y \in G\),

---

Visit IJSRTM at [https://mgesjournals.com/ijsrtm/](https://mgesjournals.com/ijsrtm/)
Let \( (xy) \) \( A \mu \leq (x) \lor A(y) \lor \lambda((x^{-1}) A \mu \leq A(x) \lor \lambda \)

Clearly, a \((0, 1)\)-anti fuzzy subgroup is just an anti fuzzy subgroup, and thus a \((\lambda, \mu)\)-anti fuzzy subgroup is a generalization of fuzzy subgroup.

**MAIN RESULTS**

**Definition:** 3.1

Let \( A \) be a fuzzy subset of \( G \). Then a \((\lambda, \mu)\)-anti fuzzy subset \( A^{(\lambda,\mu)} \) of a fuzzy set \( A \) of \( G \) is defined as, \( A^{(\lambda,\mu)} = (x, \{ A A (1 - \lambda) \} \lor (1 - \mu) : x \in G) \).

**Definition:** 3.2

Let \( A \) be a multi fuzzy subset of \( G \). Then a \((\lambda, \mu)\)-multi anti fuzzy subset \( A^{(\lambda,\mu)} \) of a multifuzzy set \( A \) of \( G \) is defined as, \( A^{(\lambda,\mu)} = (x, \{ A_i A (1 - \lambda_i) \} \lor (1 - \mu_i) : x \in G) \).

Clearly, a \((0, 1)\)-multi anti fuzzy subset is just a multi fuzzy subset of \( G \), and thus a \((\lambda, \mu)\)-multi anti fuzzy subset is also a generalization of multi fuzzy subset. Where \((0,1)\)-multi anti fuzzy subset \( A \) is defined as \( A^{(0,1)} = (A_i^{(0,1)}) \).

**Definition:** 3.3

Let \( A \) be a multi fuzzy subset of \( G \). \( A = (A_i) \) is called a \((\lambda, \mu)\)-multi anti fuzzy subgroup of \( G \)

If, for all \( x \in G, A(xy) \lor \mu \leq \max \{A(x), A(y)\} \ A \lambda_i \)

That is, \( A(xy) \lor \mu_i \leq \max \{A_i(x), A_i(y)\} \ A \lambda_i \)

Clearly, a \((0, 1)\)-multi anti fuzzy subgroup is just a multi anti fuzzy subgroup of \( G \), and thus \((\lambda, \mu)\)-multi fuzzy subgroup is also a generalization of multi fuzzy subgroup.

**Definition:** 3.4

Let \((\lambda,\mu)\) and \(B^{(\lambda,\mu)}\) be any two \((\lambda, \mu)\)-multi anti fuzzy sets having the same dimension \( k \) of \( X \).

Then

\[
\begin{align*}
(i) & \quad A^{(\lambda,\mu)} \subseteq B^{(\lambda,\mu)}, \text{ iff } A^{(\lambda,\mu)}(x) \leq B^{(\lambda,\mu)}(x) \text{ for all } x \in X \\
(ii) & \quad A^{(\lambda,\mu)} = B^{(\lambda,\mu)}, \text{ iff } A^{(\lambda,\mu)}(x) = B^{(\lambda,\mu)}(x) \text{ for all } x \in X \\
(iii) & \quad A^{(\lambda,\mu)} = \{(x, 1 - A^{(\lambda,\mu)}): x \in X\} \\
(iv) & \quad A^{(\lambda,\mu)} \cap B^{(\lambda,\mu)}(x) = \{(x, A_i^{(\lambda,\mu)} \cap B_i^{(\lambda,\mu)}(x)): x \in X\},
\end{align*}
\]

where \( A^{(\lambda,\mu)} \cap B^{(\lambda,\mu)}(x) = \min \{A_i^{(\lambda,\mu)}(x), B_i^{(\lambda,\mu)}(x)\} = \min \{A_i^{(\lambda,\mu)}(x), B_i^{(\lambda,\mu)}(x)\} \text{ for } i = 1, 2, \ldots, k \\
(v) & \quad A^{(\lambda,\mu)} \cup B^{(\lambda,\mu)}(x) = \{(x, A_i^{(\lambda,\mu)} \cup B_i^{(\lambda,\mu)}(x)): x \in X\},
\]

where \( A^{(\lambda,\mu)} \cup B^{(\lambda,\mu)}(x) = \max \{A_i^{(\lambda,\mu)}(x), B_i^{(\lambda,\mu)}(x)\} = \max \{A_i^{(\lambda,\mu)}(x), B_i^{(\lambda,\mu)}(x)\} \text{ for } i = 1, 2, \ldots, k \\

Here, \( A_i^{(\lambda,\mu)}(x) \) and \( B_i^{(\lambda,\mu)}(x) \) represents the corresponding \( i \text{th} \) position membership values of \( A^{(\lambda,\mu)} \) and \( B^{(\lambda,\mu)} \).

**Definition:** 3.5

Let \( A^{(\lambda,\mu)} = \{(x, A_i^{(\lambda,\mu)}(x)): x \in X\} \) be a \((\lambda, \mu)\)-MAFS of dimension \( k \) and let \( a = (a_1, a_2, \ldots, a_k) \in [0,1]^k \), where each \( a_i \in [0,1] \) for all i. Then the \( \alpha \) - lowerer cut of \((\lambda,\mu)\) is the set of all \( x \) such that \( A_i^{(\lambda,\mu)}(x) \leq a_i \), \( \forall i \) and is denoted by \([A^{(\lambda,\mu)}]_{\alpha_i} \). Clearly it is a crisp set.

**Definition:** 3.6

Let \( A^{(\lambda,\mu)} = \{(x, A_i^{(\lambda,\mu)}(x)): x \in X\} \) be a \((\lambda, \mu)\)-MAFS of dimension \( k \) and let \( a = (a_1, a_2, \ldots, a_k) \in [0,1]^k \), where each \( a_i \in [0,1] \) for all i. Then the \( \alpha \) - lowerer cut of \((\lambda,\mu)\) is the set of all \( x \) such that \( A_i^{(\lambda,\mu)}(x) < a_i, \forall i \) and is denoted by \([A^{(\lambda,\mu)}]_{\alpha_i} \). Clearly it is also a crisp set.

**Theorem:** 3.7 ([Feng, Y. and Yao, B. (2012)])

Let \( A \) and \( B \) are any two \((\lambda, \mu)\)-MAFSs of dimension \( k \) taken from a non-empty set \( X \). Then \( A \subseteq B \) if and only if \([A^{(\lambda,\mu)}]_{\alpha_i} \subseteq [B^{(\lambda,\mu)}]_{\alpha_i} \) for every \( \alpha_i \in [0,1]^k \).
Definition: 3.8
A MFS $A = \{ (x,A(x)): x \in X \}$ of a group $G$ is said to be a $(\lambda,\mu)$-multi anti fuzzy sub group of $G((\lambda,\mu)$-MAFSG), if it satisfies the following: For $\lambda,\mu \in [0,1]^k$, $0 \leq \lambda_i \leq \mu_i \leq 1$, $0 \leq \lambda + \mu \leq 1(\lambda) A(xy) A \mu \leq \max(A(x),A(y)) \lor \lambda$

(ii) $\langle x \rangle A \mu \leq \langle x \rangle \lor \lambda$ for all $x, y \in G$. That is,

(i) $\langle xy \rangle A \mu \leq \max\{A(x),A(y)\} \lor \lambda_i$

(ii) $\langle x \rangle A \mu_i \leq A(x) \lor \lambda_i$ for all $x, y \in G$.

Clearly, a $(0,1)$-multi anti fuzzy subgroup is just a multi anti fuzzy subgroup of $G$, and thus a $(\lambda,\mu)$-multi anti fuzzy subgroup is a generalization of multi anti fuzzy subgroup.

(i) If $A$ is a $(\lambda,\mu)$-MAFSG of $G$, then the complement of $A$ need not be an $(\lambda,\mu)$-MAFSG of $G$.

(ii) $A$ is a MAFSG of a group $\iff$ each $(\lambda,\mu)$-AFS $^{(1,\mu)}$ is a $(\lambda,\mu)$-AFSG of $G$, $i=1,2,..,k$

Definition: 3.10 (Muthuraj, R. and Balamurugan, S.(2013))
A $(\lambda,\mu)$-MAFSG $^{(1)}$ of a group $G$ is said to be a $(\lambda,\mu)$-multi anti fuzzy normal subgroup $((\lambda,\mu)$-MAFNSG) of $G$, it satisfies $A^{(1,\mu)}(x) = A^{(1,\mu)}(y)$ for all $x, y \in G$.

Definition: 3.11
Let $(G, \cdot)$ be a Groupoid and $A^{(1,\mu)}, B^{(1,\mu)}$ be any two $(\lambda,\mu)$-MAFSs having the same dimension $k$ of $G$. Then the product of $(1,\mu)$ and $B^{(1,\mu)}$, denoted by $A^{(1,\mu)} \circ B^{(1,\mu)}$ and is defined as:

$A^{(1,\mu)} \circ B^{(1,\mu)}(x) = \max\{\min\{A^{(1,\mu)}(y), B^{(1,\mu)}(z)\}: yz = x, \forall y, z \in G\}$, $\forall x \in G$

Where $A^{(1,\mu)} = (\lambda_k, \lambda, \ldots, \lambda_k \times), if x is not expressible as x = yz$

That is, $\forall x \in G,$

$A^{(1,\mu)} \circ B^{(1,\mu)}(x) = \{\max\{\min\{A^{(1,\mu)}(y), B^{(1,\mu)}(z)\}: yz = x, \forall y, z \in G\}\}$

$(\lambda_k), if x is not expressible as x = yz$

Definition: 3.12
Let $X$ and $Y$ be any two non-empty sets and $f: X \rightarrow Y$ be a mapping. Let $(1,\mu)$ and $B^{(1,\mu)}$ be any two $(\lambda,\mu)$-MAFSs of $X$ and $Y$ respectively having the same dimension $k$. Then the image of $(1,\mu)$ under the map $f$ is denoted by $f(A^{(1,\mu)})$, is defined as: $\forall y \in Y,$

$f^{(1,\mu)}(y) = \max\{A^{(1,\mu)}(x)\}: x \in f^{-1}(y)$

Also, the pre-image of $B^{(1,\mu)}$ under the map $f$ is denoted by $f^{-1}(B^{(1,\mu)})$ and is defined as: $f^{-1}(B^{(1,\mu)})(x) = (B^{(1,\mu)}(f(x)), \forall x \in X$.

Properties of $(\alpha, Q)$-lower cuts of the $(\lambda,\mu)$-MAFSG’s of a group
In this section, we have proved some theorems on $(\lambda,\mu)$-MAFSG’s of a group $G$ by using some of their $(\alpha, \beta)$-Lower cuts.

Proposition: 4.1
If $(\lambda)$ and $B^{(1,\mu)}$ are any two $(\lambda,\mu)$-MAFSs of a universal set $X$.

Then the following holds good:

(i) $[A^{(1,\mu)}]_{[\lambda,\mu]} \subseteq [\lambda,\mu]_\beta$ if $\alpha < \delta$

(ii) $A^{(1,\mu)} \subseteq B^{(1,\mu)}$ implies $[B^{(1,\mu)}]_{\lambda_{\alpha}} \subseteq [A^{(1,\mu)}]_{\alpha}$

(iii) $[A^{(1,\mu)}] \cap B^{(1,\mu)} [\lambda_{\alpha}]_{\delta} \subseteq [A^{(1,\mu)}] \cap [B^{(1,\mu)}]_{\lambda_{\alpha}}$

(iv) $[A^{(1,\mu)}] \cup B^{(1,\mu)} \subseteq [A^{(1,\mu)}] \cup [B^{(1,\mu)}]$ (here equality holds if $\alpha = 1, \forall i) \cup [\lambda]_{\alpha} [\lambda_{\alpha}]_{\delta} \in [0,1]_{\delta}$
Proposition: 4.2
Let $(G,.)$ be a groupoid and $A^{(k)}_{\mu}$ and $B^{(k)}_{\mu}$ are any two $(\lambda, \mu)$ − MAFS’s of $G$. Then we have

$$[A^{(k)}_{\mu} \circ B^{(k)}_{\mu}]_{\alpha} = [A^{(k)}_{\mu}]_{\alpha} [B^{(k)}_{\mu}]_{\alpha}, \text{ where } \alpha \in [0,1]^k.$$

Theorem: 4.3
If $(k)$ is a $(\lambda, \mu)$-multi anti fuzzy subgroup of $G$ and $\alpha \in [0,1]^k$, then the $\alpha − \text{lower cut}$ of $(k)$, $[A^{(k)}_{\mu}]_{\alpha}$ is a subgroup of $G$, where $A^{(k)}_{\mu}(e) \leq \alpha$ and ‘e’ is the identity element of $G$.

Proof:
We have, $(k)(e) \leq \alpha, e \in [A^{(k)}_{\mu}]_{\alpha}$. Therefore $(k)_{\alpha} \neq \emptyset$.

Let $x, y \in [k]_{\alpha}$. Then $(k)(x) \leq \alpha$ and $A^{(k)}_{\mu}(y) \leq \alpha$.

Then for all $i$, $(k)_{\alpha(i)}(x) \leq \alpha_i$ and $A^{(k)}_{\mu}(y) \leq \alpha_i$.

$$\Rightarrow \max\{A^{(k)}_{\mu}(x), A^{(k)}_{\mu}(y)\} \leq \alpha_i, \forall \ i \ldots \ldots \ldots \ (1)$$

$$\Rightarrow A^{(k)}_{\mu}(xy^{-1}) \leq \max\{A^{(k)}_{\mu}(x), A^{(k)}_{\mu}(y)\} \leq \alpha_i, \forall \ i$$

since $A^{(k)}_{\mu}$ is a $(\lambda, \mu)$-multi anti fuzzy subgroup of a group $G$ and by (1).

$$\Rightarrow (k)_{\alpha(i)}(xy^{-1}) \leq \alpha_i, \forall \ i$$

$$\Rightarrow 1 \leq \alpha$$

$$\Rightarrow xy^{-1} \in [k]_{\alpha}$$

$$\Rightarrow [k]_{\alpha} \text{ is a subgroup of } G.$$
Proof:
Let \( x \in [x]_\alpha \) and \( g \in G \). Then \( (\langle i \rangle_\alpha)(e) \leq \alpha \).
That is, \( (\langle i \rangle_{\lambda_\mu})(x) \leq \alpha_i \) , \( \forall i \).
Since \( [x]_\alpha \) is a \((\lambda, \mu)\)-MAFNSG of \( G \),
\( (\langle i \rangle_{\lambda_\mu})(g^{-1} x g) = A_{\langle i \rangle_{\lambda_\mu}}(x) \), \( \forall i \).
\( \Rightarrow (\langle i \rangle_{\lambda_\mu})(g^{-1} x g) = A_{\langle i \rangle_{\lambda_\mu}}(x) \leq \alpha_i \), and \( \forall i \), by using (1).
\( \Rightarrow (\langle i \rangle_{\lambda_\mu})(g^{-1} x g) \leq \alpha_i \ \forall i \)
\( \Rightarrow (\langle i \rangle_{\lambda_\mu})(g^{-1} x g) \leq \alpha \Rightarrow g^{-1} x g \in [A_{\langle i \rangle_{\lambda_\mu}}]_\alpha \)
\( \Rightarrow [x]_\alpha \) is normal subgroup of \( G \).

**Theorem: 4.6**

If \( A_{\langle i \rangle_{\lambda_\mu}} \) and \( B_{\langle i \rangle_{\lambda_\mu}} \) are any two \((\lambda, \mu)\)-multi anti fuzzy subgroups ((\(\lambda, \mu\))-MAFSGs) of a group \( G \), then \( A_{\langle i \rangle_{\lambda_\mu}} \cup B_{\langle i \rangle_{\lambda_\mu}} \) is also a \((\lambda, \mu)\)-multi anti fuzzy subgroup of \( G \).

**Proof:**
\[ (\langle \lambda_\mu \rangle, \langle \lambda_\mu \rangle^{-1}) = A_{\langle \lambda_\mu \rangle} \]

Assume \( A_{\langle \lambda_\mu \rangle} \) and \( B_{\langle \lambda_\mu \rangle} \) are any two \((\lambda, \mu)\)-multi anti fuzzy subgroup of a group \( G \), then \( \forall x, y \in G \),
(i) \( (\langle \lambda_\mu \rangle)(xy^{-1}) \leq \max\{A_{\langle \lambda_\mu \rangle}(x), A_{\langle \lambda_\mu \rangle}(y)\} \)

(ii) \( B_{\langle \lambda_\mu \rangle}(xy^{-1}) \leq \max\{B_{\langle \lambda_\mu \rangle}(x), B_{\langle \lambda_\mu \rangle}(y)\} \).

Then \( A_{\langle \lambda_\mu \rangle} \cup B_{\langle \lambda_\mu \rangle} \) is a \((\lambda, \mu)\)-multi antifuzzy subgroup of \( G \).

**Remark: 4.7**
The intersection of two \((\lambda, \mu)\)-multi antifuzzy subgroups of a group \( G \) need not be a \((\lambda, \mu)\)-MAFSG of the group \( G \).

**Proof:**
Consider the Klein’s four group \( G = \{a, b, \bar{a}, \bar{b}\} \), where \( a^2 = e = b^2 \) and \( ba = ab \).
Let \( \forall i \leq 4 \), \( \forall s_i \in [0,1]^k \) such that
\( r_0 \leq r_1 \leq \ldots \leq r_5 \) and \( s_0 \leq s_1 \leq r_5 \).
Define \( \lambda_\mu \) – MAFSG \( A_{\langle \lambda_\mu \rangle} \) and \( B_{\langle \lambda_\mu \rangle} \) of dimension \( k \) as follows:
\( A_{\langle \lambda_\mu \rangle} = \{x, A_{\langle \lambda_\mu \rangle}(x) \in G \} \)
\( B_{\langle \lambda_\mu \rangle} = \{x, B_{\langle \lambda_\mu \rangle}(x) \in G \} \),
where \( A_{\langle \lambda_\mu \rangle}(e) = r_0 \) \((1 - \lambda_\mu) \vee (1 - \mu)\), \( A_{\langle \lambda_\mu \rangle}(a) = r_3 \) \((1 - \lambda_\mu) \vee (1 - \mu)\), \( A_{\langle \lambda_\mu \rangle}(b) = r_4 \) \((1 - \lambda_\mu) \vee (1 - \mu)\), \( A_{\langle \lambda_\mu \rangle}(ab) = r_0 \) \((1 - \lambda_\mu) \vee (1 - \mu)\).

Clearly \((\lambda_\mu)\) and \( B_{\langle \lambda_\mu \rangle} \) are \((\lambda, \mu)\)-multi anti fuzzy subgroups of \( G \).

Now \( A_{\langle \lambda_\mu \rangle} \cap B_{\langle \lambda_\mu \rangle} = \{x, A_{\langle \lambda_\mu \rangle}(x) \in G \} \), where
\( (A_{\lambda_\mu}(\langle \lambda_\mu \rangle, B_{\lambda_\mu}(\langle \lambda_\mu \rangle)) = \min\{A_{\lambda_\mu}(\langle \lambda_\mu \rangle, B_{\lambda_\mu}(\langle \lambda_\mu \rangle)\} \)

\( (\lambda_\mu) \cap B_{\lambda_\mu}(\langle \lambda_\mu \rangle)(e) = r_0 \) \((1 - \lambda_\mu) \vee (1 - \mu)\), \( A_{\lambda_\mu}(\langle \lambda_\mu \rangle)(a) = r_3 \) \((1 - \lambda_\mu) \vee (1 - \mu)\), \( A_{\lambda_\mu}(\langle \lambda_\mu \rangle)(b) = r_4 \) \((1 - \lambda_\mu) \vee (1 - \mu)\), \( A_{\lambda_\mu}(\langle \lambda_\mu \rangle)(ab) = r_0 \) \((1 - \lambda_\mu) \vee (1 - \mu)\).

**Remark: 4.7**
The intersection of two \((\lambda, \mu)\)-multi antifuzzy subgroups of a group \( G \) need not be a \((\lambda, \mu)\)-MAFSG of the group \( G \).

**Proof:**
Consider the Klein’s four group \( G = \{e, a, b, \bar{a}, \bar{b}\} \), where \( a^2 = e = b^2 \) and \( ba = ab \).
Let \( \forall i \leq 4 \), \( \forall s_i \in [0,1]^k \) such that
\( r_0 \leq r_1 \leq \ldots \leq r_5 \) and \( s_0 \leq s_1 \leq s_6 \).
Define \( \lambda_\mu \) – MAFSG \( A_{\lambda_\mu} \) and \( B_{\lambda_\mu} \) of dimension \( k \) as follows:
\( A_{\lambda_\mu} = \{x, A_{\lambda_\mu}(x) \in G \} \)
\( B_{\lambda_\mu} = \{x, B_{\lambda_\mu}(x) \in G \} \),
where \( A_{\lambda_\mu}(e) = r_0 \) \((1 - \lambda_\mu) \vee (1 - \mu)\), \( A_{\lambda_\mu}(a) = r_3 \) \((1 - \lambda_\mu) \vee (1 - \mu)\), \( A_{\lambda_\mu}(b) = r_4 \) \((1 - \lambda_\mu) \vee (1 - \mu)\), \( A_{\lambda_\mu}(ab) = r_0 \) \((1 - \lambda_\mu) \vee (1 - \mu)\).

Clearly \((\lambda_\mu)\) and \( B_{\lambda_\mu} \) are \((\lambda, \mu)\)-multi anti fuzzy subgroups of \( G \).

Now \( A_{\lambda_\mu} \cap B_{\lambda_\mu} = \{x, A_{\lambda_\mu}(x) \in G \} \), where
\( (A_{\lambda_\mu}(\lambda_\mu), B_{\lambda_\mu}(\lambda_\mu)) = \min\{A_{\lambda_\mu}(\lambda_\mu), B_{\lambda_\mu}(\lambda_\mu)\} \)

\( (\lambda_\mu) \cap B_{\lambda_\mu}(\lambda_\mu)(e) = r_0 \) \((1 - \lambda_\mu) \vee (1 - \mu)\), \( A_{\lambda_\mu}(\lambda_\mu)(a) = r_3 \) \((1 - \lambda_\mu) \vee (1 - \mu)\), \( A_{\lambda_\mu}(\lambda_\mu)(b) = r_4 \) \((1 - \lambda_\mu) \vee (1 - \mu)\), \( A_{\lambda_\mu}(\lambda_\mu)(ab) = r_0 \) \((1 - \lambda_\mu) \vee (1 - \mu)\).

**Remark: 4.7**
The intersection of two \((\lambda, \mu)\)-multi antifuzzy subgroups of a group \( G \) need not be a \((\lambda, \mu)\)-MAFSG of the group \( G \).
Since \([e, a, b]\) is not a subgroup of \(G\), \([A^{(1)}_{\lambda}] \cup B^{(2)}_{\mu}\) is not a subgroup of \(G\). Hence \([A^{(1)}_{\lambda}] \cup B^{(2)}_{\mu}\) is not a subgroup of \(G\) and there fore \([A^{(1)}_{\lambda}] \cup B^{(2)}_{\mu}\) is not a \((\lambda, \mu)\)-MAFSG of the group \(G\).

**Example:** 4.8

There are two cases needed to clarify the previous theorem 3.7 and remark.

**Case (i):** Consider the abelian group \(G = \{e, a, b, c\}\) with usual multiplication such that \(a^2 = e = b^2\) and \(ab = ba\).

Let \(A^{(1)}_{\lambda} = \{e, 0.3 A (1 - \lambda_1) V (1 - \mu_1), 0.2 A (1 - \lambda_2) V (1 - \mu_2)\}, a, 0.5 A (1 - \lambda_1) V (1 - \mu_1), 0.6 A (1 - \lambda_2) V (1 - \mu_2)\}, <b, 0.6 A (1 - \lambda_1) V (1 - \mu_1), 0.6 A (1 - \lambda_2) V (1 - \mu_2)\}\) and \(B^{(2)}_{\mu} = \{e, 0.2 A (1 - \lambda_1) V (1 - \mu_1), 0.3 A (1 - \lambda_2) V (1 - \mu_2)\}, a, 0.7 A (1 - \lambda_1) V (1 - \mu_1), 0.8 A (1 - \lambda_2) V (1 - \mu_2)\}\) and \(A^{(1)}_{\lambda} \cup B^{(2)}_{\mu} = \{e, 0.3 A (1 - \lambda_1) V (1 - \mu_1), 0.3 A (1 - \lambda_2) V (1 - \mu_2)\}, a, 0.5 A (1 - \lambda_1) V (1 - \mu_1), 0.6 A (1 - \lambda_2) V (1 - \mu_2)\}\) and \(A^{(1)}_{\lambda} \cap B^{(2)}_{\mu} = \{e, 0.2 A (1 - \lambda_1) V (1 - \mu_1), 0.2 A (1 - \lambda_2) V (1 - \mu_2)\}, <a, 0.4 A (1 - \lambda_1) V (1 - \mu_1), 0.6 A (1 - \lambda_2) V (1 - \mu_2)\}\) be two \((\lambda, \mu)\)-MAFSs having dimension two of \(G\). Clearly \((\lambda)\) and \(B^{(2)}_{\mu}\) are \((\lambda, \mu)\)-MAFSGs of \(G\).

Then \(A^{(1)}_{\lambda} \cup B^{(2)}_{\mu}\) be a \((\lambda, \mu)\) \(-\) MAFS of \(G\) and \(A^{(1)}_{\lambda} \cap B^{(2)}_{\mu}\) is not a \((\lambda, \mu)\)-MAFSG of \(G\). Hence \(ca(i)\).

**Case (ii):** Consider the abelian group \(G = \{e, a, b, ab\}\) with usual multiplication such that \(a^2 = e = b^2\) and \(ab = ba\).

Let \(A^{(1)}_{\lambda} = \{e, 0.3 A (1 - \lambda_1) V (1 - \mu_1), 0.1 A (1 - \lambda_2) V (1 - \mu_2)\}, <a, 0.1 A (1 - \lambda_1) V (1 - \mu_1), 0.4 A (1 - \lambda_2) V (1 - \mu_2)\}\) and \(B^{(2)}_{\mu} = \{e, 0.2 A (1 - \lambda_1) V (1 - \mu_1), 0.2 A (1 - \lambda_2) V (1 - \mu_2)\}, <a, 0.4 A (1 - \lambda_1) V (1 - \mu_1), 0.6 A (1 - \lambda_2) V (1 - \mu_2)\}\) be two \((\lambda, \mu)\)-MAFSs having dimension two of \(G\). Clearly \((\lambda)\) and \(B^{(2)}_{\mu}\) are \((\lambda, \mu)\)-MAFSGs of \(G\).

Then \(A^{(1)}_{\lambda} \cup B^{(2)}_{\mu}\) be a \((\lambda, \mu)\) \(-\) MAFSG of \(G\). Let \(G\) be a group and \((\lambda)\) be a \((\lambda, \mu)\)-MAFSG of \(G\). Let \(x \in G\) be a fixed element. Then the set \(A^{(1)}_{\lambda}(x) = A^{(1)}_{\lambda}(x^{-1})\) of all \((\lambda, \mu)\)-multi anti fuzzy left coset of \(G\) determined by \(A^{(1)}_{\lambda}\) and \(x\).

Similarly , the set \(A^{(1)}_{\lambda}(g) = A^{(1)}_{\lambda}(g^{-1}), \forall g \in G\) is called the \((\lambda, \mu)\)-multi anti fuzzy right coset of \(G\) determined by \(A^{(1)}_{\lambda}\) and \(x\).

**Remark:** 5.2

It is clear that if \((\lambda)\) is a \((\lambda, \mu)\)-multi anti fuzzy normal subgroup of \(G\), then \((\lambda, \mu)\)-multi anti fuzzy left coset and the \((\lambda, \mu)\)-multi anti fuzzy right coset of \(A^{(1)}_{\lambda}\) on \(G\) coincides and in this case, we simply call it as \((\lambda, \mu)\)-multi anti fuzzy coset.

**Example:** 5.3

Let \(G\) be a group. Then \(A^{(1)}_{\lambda} = \{x, A^{(1)}_{\lambda}(x) : x \in G/A^{(1)}_{\lambda}\} = A^{(1)}_{\lambda}(e)\} is a \((\lambda, \mu)\)-multi anti fuzzy normal subgroup of \(G\).

**Theorem:** 5.4

Let \((\lambda)\) be a \((\lambda, \mu)\)-multi anti fuzzy subgroup of \(G\) and \(x\) be any fixed element of \(G\). Then the following holds:

**(i)** \(x[A^{(1)}_{\lambda}(x)]_{\lambda} = x A^{(1)}_{\lambda}(x)_{\lambda}\)
(ii) \([A^{\lambda,\mu}]_\alpha x = \{[x,\mu]_\alpha\}, \forall \alpha \in [0,1]^k\) with \(0 \leq \alpha_i \leq 1, \forall i.

**Proof:**

(i) \([x \cdot A^{[k,\mu]}]_\alpha = \{g \in G : A^{[k,\mu]}(x \cdot g) \leq \alpha\} \) with \(0 \leq \alpha_i \leq 1, \forall i.\) Also \(x \cdot A^{[k,\mu]}_\alpha = x\{y \in G : \lambda(y) \leq \alpha_i\} = \{xy \in G : A^{[k,\mu]}(y) \leq \alpha\} \)

Put \(xy = g \Rightarrow y = x^{-1}g.\) Then (1) can be written as,

\(x \cdot A^{[k,\mu]}_\alpha = \{g \in G : A^{[k,\mu]}(x^{-1}g) \leq \alpha\} = \{g \in G : x \cdot A^{[k,\mu]}(g) \leq \alpha\} = [x \cdot A^{[k,\mu]}]_\alpha \)

Therefore, \([A^{[k,\mu]}]_\alpha \in [A^{[k,\mu]}]_\alpha, \forall \alpha \in [0,1]^k \) with \(0 \leq \alpha_i \leq 1, \forall i.\)

(ii) Now \([A^{[k,\mu]}]_\alpha = \{g \in G : A^{[k,\mu]}(g) \leq \alpha\} \) with \(0 \leq \alpha_i \leq 1, \forall i.\) Also \([A^{[k,\mu]}]_\alpha x = \{y \in G : A^{[k,\mu]}(y) \leq \alpha \leq \beta\}x = \{yx \in G : A^{[k,\mu]}(yx) \leq \alpha\} \)

Set \(yx = g \Rightarrow y = gx^{-1}.\) Then (2) can be written as \([A^{[k,\mu]}]_\alpha x = \{g \in G : A^{[k,\mu]}(g) \leq \alpha\} = \{g \in G : A^{[k,\mu]}(x \cdot g) \geq \alpha\} = [A^{[k,\mu]}]_\alpha \)

Therefore, \([A^{[k,\mu]}]_\alpha x = [A^{[k,\mu]}]_\alpha, \forall \alpha \in [0,1]^k \) with \(0 \leq \alpha_i \leq 1, \forall i.\)

**Homomorphisms of \((\lambda, \mu)\) – Multi fuzzy subgroup**

In this section, we shall prove some theorems on \((\lambda, \mu)\) – MAFSG’s of a group byhomomorphism.

**Preposition: 6.1**

Let \(f: X \rightarrow Y\) be an onto map. If \((\lambda, \mu)\) and \((B^{[k,\mu]}_\alpha)\) are two \((\lambda, \mu)\)–multi anti fuzzy sets of multifuzzy sets A and B with dimension k of X and Y respectively, then the following hold:

(i) \(f(A^{[k,\mu]}_\alpha) \subseteq \{f(A^{[k,\mu]}_\alpha)\}_\alpha \)

(ii) \(f^{-1}(B^{[k,\mu]}_\alpha) = \{f^{-1}(B^{[k,\mu]}_\alpha)\}_\alpha, \forall \alpha \in [0,1]^k \) with \(0 \leq \alpha_i \leq 1, \forall i.\)

**Proof:** (i) Let \(y \in f(A^{[k,\mu]}_\alpha)\). Then there exist an element \(x \in [A^{[k,\mu]}_\alpha] \) such that \(f(x) = y\). Then we have \([x,\mu]_\alpha \leq \alpha,\)

Since \(x \in [A^{[k,\mu]}_\alpha] \)
\(\Rightarrow [x,\mu]_\alpha \leq \alpha \)
\(\Rightarrow \min[A^{[k,\mu]}_\alpha](x) \leq \alpha \)
\(\Rightarrow \min[A^{[k,\mu]}_\alpha](x) \leq \alpha \)
\(\Rightarrow f(A^{[k,\mu]}_\alpha) \subseteq \{f(A^{[k,\mu]}_\alpha)\}_\alpha \)

Therefore, \((A^{[k,\mu]}_\alpha) \subseteq \{f(A^{[k,\mu]}_\alpha)\}_\alpha, \forall \alpha \in [0,1]^k \) with \(0 \leq \alpha_i \leq 1, \forall i.\)

(ii) Let \(x \in [f^{-1}(B^{[k,\mu]}_\alpha)] \Rightarrow [x \in X : f^{-1}(B^{[k,\mu]}_\alpha)(x) \leq \alpha \)

\[- [x \in X : f^{-1}(B^{[k,\mu]}_\alpha)(x) \leq \alpha] \), \forall i.\)
\[- [x \in X : f^{-1}(B^{[k,\mu]}_\alpha)(x) \leq \alpha] \), \forall i.\)
\[- [x \in X : f^{-1}(B^{[k,\mu]}_\alpha)(x) \leq \alpha] \), \forall i.\)
\[- [x \in X : f^{-1}(B^{[k,\mu]}_\alpha)(x) \leq \alpha] \), \forall i.\)
\[- [x \in X : f^{-1}(B^{[k,\mu]}_\alpha)(x) \leq \alpha] \)

**Theorem: 6.2**

Let \(f: G_1 \rightarrow G_2\) be an onto homomorphism and if \((\lambda, \mu)\) is a \((\lambda, \mu)\)–MAFSG of G1, then

\((\lambda, \mu)\) is a \((\lambda, \mu)\)–MAFSG of group G2.

**Proof:**

By theorem 4.4, it is enough to prove that each \((\alpha, \beta)\) – lower cuts \([\lambda, \mu]_\alpha\) is a subgroup of \(G_2, \forall \alpha \in [0,1]^k \) with \(0 \leq \alpha_i \leq 1, \forall i.\). Let \(y_1, y_2 \in [\lambda, \mu]_\alpha\). \(\exists (\alpha_i, \mu_i) \)

Then \((\lambda, \mu)(y_1) \leq \alpha, \) and \(f(A^{[k,\mu]}_\alpha)(y_2) \leq \alpha \)
\(\Rightarrow (A^{[k,\mu]}_\alpha)(y_1) \leq \alpha, \) and \(f(A^{[k,\mu]}_\alpha)(y_2) \leq \alpha, \forall i \)
By the proposition 6.1(i), we have \( f((A^{(i)}_{\lambda,\mu})_a) \subseteq f(A^{(i)}_{\lambda,\mu})_a \), \( \forall A^{(i)}_{\lambda,\mu} \in (\lambda, \mu) - MAFSG(G_1) \).

Since \( f \) is onto, there exists some \( x_1 \) and \( x_2 \) in \( G_1 \) such that \( f(x_1) = y_1 \) and \( f(x_2) = y_2 \). Therefore, (1) can be written as \( f(A^{(i)}_{\lambda,\mu})((x_1)) \leq \alpha, \text{ and } f(A^{(i)}_{\lambda,\mu})((x_2)) \leq \alpha, \forall i \).

\( \Rightarrow A^{(i)}_{\lambda,\mu}(x_1) \leq f(A^{(i)}_{\lambda,\mu})(f(x_1)) \leq \alpha, \text{ and } A^{(i)}_{\lambda,\mu}(x_2) \leq f(A^{(i)}_{\lambda,\mu})(f(x_2)) \leq \alpha, \forall i.\)

\( \Rightarrow A^{(i)}_{\lambda,\mu}(x_1) \leq \alpha, \text{ and } A^{(i)}_{\lambda,\mu}(x_2) \leq \alpha, \forall i.\)

\( \Rightarrow \max\{A^{(i)}_{\lambda,\mu}(x_1), A^{(i)}_{\lambda,\mu}(x_2)\} \leq \alpha.\)

\( A^{(i)}_{\lambda,\mu}(x_2^{-1}) \leq \max\{A^{(i)}_{\lambda,\mu}(x_1), A^{(i)}_{\lambda,\mu}(x_2)\}, \text{ since } A^{(i)}_{\lambda,\mu} \in (\lambda, \mu) - MAFSG(G_1).\)

\( \Rightarrow (\lambda, \mu)(x_1x_2^{-1}) \leq \alpha \)

\( \Rightarrow x_1x_2^{-1} \in [A^{(i)}_{\lambda,\mu}], \Rightarrow f(x_1x_2^{-1}) \in f([A^{(i)}_{\lambda,\mu}]) \subseteq [f(A^{(i)}_{\lambda,\mu})]_a \)

\( \Rightarrow f(x_1)f(x_2^{-1}) \in [f(A^{(i)}_{\lambda,\mu})] \Rightarrow f(x_1)f(x_2^{-1}) \in [f(A^{(i)}_{\lambda,\mu})] \Rightarrow y_1y_2^{-1} \in [f(A^{(i)}_{\lambda,\mu})]_a \)

\( \Rightarrow [f(A^{(i)}_{\lambda,\mu})]_a \text{ is a subgroup of } G_2, \forall \alpha \in [0,1] \Rightarrow f(A^{(i)}_{\lambda,\mu}) \in (\lambda, \mu) - MAFSG(G_2).\)

Corollary: 6.3

If \( f: G_1 \rightarrow G_2 \) be a homomorphism of a group \( G_1 \) onto a group \( G_2 \) and \( \{ A^{(i)}_{\lambda,\mu} : i \in I \} \) be a family of \( (\lambda, \mu) - MAFSGs \) of \( G_1 \), then \( f(U A^{(i)}_{\lambda,\mu}) \) is an \( (\lambda, \mu) - MAFSG \) of \( G_2 \).

**Theorem: 6.4**

Let \( f: G_1 \rightarrow G_2 \) be a homomorphism of a group \( G_1 \) into a group \( G_2 \). If \( A^{(i)}_{\lambda,\mu} \) is an \((\lambda, \mu) - MAFSG \) of \( G_2 \), then \( f^{-1}(B^{(i)}_{\lambda,\mu}) \) is also a \((\lambda, \mu) - MAFSG \) of \( G_1 \).

**Proof:**

By theorem 4.4, it is enough to prove that \( f^{-1}(B^{(i)}_{\lambda,\mu}) \) is a subgroup of \( G_1 \), with \( 0 \leq \alpha, \leq 1, \forall i.\)

Let \( x_1, x_2 \in [f^{-1}(B^{(i)}_{\lambda,\mu})]_a \). Then \( f^{-1}(B^{(i)}_{\lambda,\mu})(x_1) \leq \alpha \) and \( f^{-1}(B^{(i)}_{\lambda,\mu})(x_2) \leq \alpha \Rightarrow B^{(i)}_{\lambda,\mu}(f(x_1)) \leq \alpha \) and \( B^{(i)}_{\lambda,\mu}(f(x_2)) \leq \alpha \)

\( \Rightarrow \max\{B^{(i)}_{\lambda,\mu}(f(x_1)), B^{(i)}_{\lambda,\mu}(f(x_2))\} \leq \alpha \)

\( B^{(i)}_{\lambda,\mu}(f(x_1))x_2^{-1} \leq \max\{B^{(i)}_{\lambda,\mu}(f(x_1)), B^{(i)}_{\lambda,\mu}(f(x_2))\} \leq \alpha, \text{ since } B^{(i)}_{\lambda,\mu} \in (\lambda, \mu) - MAFSG(G_2).\)

\( \Rightarrow f(x_1)f(x_2^{-1}) \in [B^{(i)}_{\lambda,\mu} \Rightarrow f(x_1)x_2^{-1} \in [B^{(i)}_{\lambda,\mu}], \text{ since } f \text{ is homomorphism}.\)

\( \Rightarrow x_1x_2^{-1} \in f^{-1}([B^{(i)}_{\lambda,\mu}]_a) = [f^{-1}(B^{(i)}_{\lambda,\mu})]_a, \text{ by the proposition 6.1(ii)}.\)

\( \Rightarrow x_1x_2^{-1} \in [f^{-1}(B^{(i)}_{\lambda,\mu})]_a \Rightarrow [f^{-1}(B^{(i)}_{\lambda,\mu})]_a \text{ is a subgroup of } G_1.\)

\( \Rightarrow f^{-1}(B^{(i)}_{\lambda,\mu}) \text{ is a } (\lambda, \mu) - MAFSG \text{ of } G_1.\)

**Theorem: 6.5**

Let \( f: G_1 \rightarrow G_2 \) be a surjective homomorphism and if \( A^{(i)}_{\lambda,\mu} \) is a \((\lambda, \mu) - MAFSG \) of a group \( G_1 \), then \( f(A^{(i)}_{\lambda,\mu}) \) is also a \((\lambda, \mu) - MAFNSG \) of a group \( G_2 \).

**Proof:**

Let \( g_2 \in G_2 \) and \( y \in (A^{(i)}_{\lambda,\mu}). \) Since \( f \) is surjective, there exists \( g_1 \in G_1 \) and \( x \in (A^{(i)}_{\lambda,\mu}). \) such that \( f(x) = y \) and \( f(g_1) = g_2. \)

Also, since \( A^{(i)}_{\lambda,\mu} \) is a \((\lambda, \mu) - MAFNSG \) of \( G_1, A^{(i)}_{\lambda,\mu}(g_1^{-1}xg_1) = A^{(i)}_{\lambda,\mu}(x), \forall x \in A^{(i)}_{\lambda,\mu} \) and \( g_1 \in G_1.\)

Now consider, \( f(A^{(i)}_{\lambda,\mu})(g_1^{-1}xg_1) = f(A^{(i)}_{\lambda,\mu})(f(g_1^{-1}xg_1)) = f(A^{(i)}_{\lambda,\mu})(y) \), since \( f \) is a homomorphism, where \( y = f(g_1^{-1}xg_1) = g_2^{-1}yg_2 = \min\{A^{(i)}_{\lambda,\mu}(x) : x = y \text{ for } x \in G_1\} = \min\{A^{(i)}_{\lambda,\mu}(x) : f(g_1^{-1}xg_1) = y \} = \min\{A^{(i)}_{\lambda,\mu}(x) : f(g_1^{-1}xg_1) = y \} = g_1^{-1}yg_2 \text{ for } x \in A^{(i)}_{\lambda,\mu}, g_1 \in G_1 = \min\{A^{(i)}_{\lambda,\mu}(x) : f(g_1^{-1}xg_1) = y \} = g_1^{-1}yg_2 \text{ for } x \in A^{(i)}_{\lambda,\mu}, g_1 \in G_1 = \min\{A^{(i)}_{\lambda,\mu}(x) : g_2^{-1}yg_2 = g_2^{-1}yg_2 \text{ for } x \in G_1\} = \min\{A^{(i)}_{\lambda,\mu}(x) : f(x) = y \text{ for } x \in G_1\} = f(A^{(i)}_{\lambda,\mu}(y)). \) Hence \( A^{(i)}_{\lambda,\mu} \) is a \((\lambda, \mu) - MAFNSG \) of \( G_2.\)

**CONCLUSION**

In the theory of fuzzy sets, the level subsets are vital role for its development. Similarly, the \((\lambda, \mu) - \text{mutli fuzzy subgroups are very important role for the development of the theory of multi fuzzy subgroup of a group. In this paper an attempt has been made to study some algebraic natures of (\lambda, \mu) - \text{ multi anti fuzzy subgroups.} \)
REFERENCES


Visit IJSRTM at https://mgesjournals.com/ijsrtm/